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Edited by

Andrea Stupňanová, Martin Dyba, Viktor Pavliska

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Invited talks

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Katarzyna KACZMAREK-MAJER

Systems Research Institute, Polish Academy of Sciences
Warsaw, Poland

Title: **Linguistic Summaries with Semi-Supervised Fuzzy Clustering**

Currently, the potential of the data collected from sensors is only partially explored in healthcare. One of the main challenges is providing adequate explanations concerning the underlying structure of data and models. At the same time, the need for explanations is of utmost importance not only due to various regulations but also to increase trust among systems' users.

The proposed approach combines theoretical aspects of semi-supervised learning from partially-labeled sensor data with fuzzy linguistic summarization. Linguistic summarization belongs to the class of data-to-text approaches. We construct linguistic summaries for the partially labelled data streams, and the drifts in data streams are reflected in the construction of linguistic variables. The proposed approach enables to summarize of large data streams into meaningful and human-consistent information granules.

At the same time, semi-supervised fuzzy clustering is particularly promising for explaining sensor data because it enables to capture the information about the hidden structure of evolving data streams which are sparsely labelled and subject to multiple sources of uncertainty, and this was the main motivation for exploring this approach.

Semi-supervised learning is also often said to be "halfway between supervised and unsupervised learning". We will also explain and verify how to assess the impact of partial supervision properly and what are its consequences. Furthermore, in some application contexts, one can question whether all available labels are equally valid and shall be extrapolated. To alleviate the problem of misguided supervision affecting the model's performance, we discuss a regularization approach that incorporates the uncertainty into the fuzzy c-means semi-supervised learning.

Finally, we present a case study in smartphone-based mental health monitoring. Acoustic features of speech are promising as objective markers for mental health monitoring. Specialized smartphone apps can gather such acoustic data without disrupting the daily activities of patients. Nonetheless, the psychiatric assessment of the patient's mental state is typically a sporadic occurrence that takes place every few months. Consequently, only a slight fraction of the acoustic data is labelled and applicable for supervised learning. Numerical experiments for real-life and simulated data illustrate the performance of the proposed uncertainty-aware semi-supervised models.

Vicenç TORRA

Umeå university
Umeå, Sweden

Title: **Fuzzy clustering and fuzzy measures in data privacy**

Data privacy provides definitions of what privacy is, as well as methods to protect data and metrics to evaluate how much disclosure takes place for a given data release.

In this talk we will describe the usual workflow for building a data release. This consists of anonymizing or masking the data (i.e., applying a data protection mechanism), evaluating its utility, and analyzing its risk. A good masking method is one that achieves a good trade-off between risk and utility.

Then, we will show the use of fuzzy clustering in the process of data masking, and the use of fuzzy measures and metric learning to evaluate in what extent the masked data is safe and avoids identity disclosure.

Stefania BOFFA

Dipartimento di Informatica, Sistemistica e Comunicazione,
University of Milano–Bicocca, viale Sarca 336 – 20126 Milano, Italy

Title: **Intermediate quantifiers and structures of opposition in Fuzzy Formal Concept Analysis**

Formal Concept Analysis (FCA) is a mathematical theory employed for the analysis of data and classification with wide popularity in numerous application domains. FCA techniques extract special clusters called *formal concepts* from a given formal context. A formal context is a triple composed of a set of objects, a set of attributes, and a binary relation between objects and attributes. Several approaches extending FCA were developed by considering *fuzzy formal contexts* and *fuzzy formal concepts*, where attributes are satisfied by objects with truth degrees belonging to a graded scale, which is usually the real interval [0, 1]. Fuzzy formal concepts are mathematically constructed using the fuzzy quantifiers *for all* and *there exists* (the universal and existential quantifiers). We introduced a particular class of fuzzy quantifiers as new tools to capture more detailed information from datasets in FCA. These are interpretations in a model of special formulas called *intermediate quantifiers* of the formal theory of intermediate generalized quantifiers. Thus, we mainly achieved the following goals. Firstly, we proposed a novel notion of fuzzy formal concepts based on the intermediate quantifiers *almost all, most, many, few,* and *some*; moreover, we provided concrete models of graded extensions of Aristotle's square, in terms of fuzzy formal concepts. Secondly, we employed a wider class of intermediate quantifiers to extract fuzzy formal concepts from more complex datasets, which are composed of a family of formal contexts (instead of a single one) and several fuzzy relations between objects of different types.

Zdenko TAKÁČ

Institute of Information Engineering, Automation and Mathematics, Slovak University of Technology in Bratislava
Bratislava, Slovakia

Title: **Fuzzy Integrals in the Interval-Valued Setting**

In the context of integration procedures, the standard additivity of set functions was found to be rather restrictive already at the beginning of the 20th century. This limitation prompted researchers to explore set functions with more flexible forms of additivity, leading to the development of fuzzy measures, set functions without any form of additivity property. The fuzzy integral, an integral built with respect to fuzzy measures, extends the classical integral, offering a more flexible representation of uncertainty. This extension provides a systematic and mathematically rigorous framework for modeling and managing uncertainty.

One effective strategy for addressing imprecision in data aggregation involves using intervals,

where the width of the interval reflects the uncertainty associated with each object. The use of intervals has proven to be a suitable approach for handling imprecision, leading to significant efforts in developing mechanisms to fuse information in the interval-valued setting. However, a notable challenge in the aggregation of intervals lies in the absence of an natural/intuitive total order, particularly for functions where the total order is an essential component, such as fuzzy integrals.

This presentation explores various approaches to define and handle fuzzy integrals within the interval-valued setting, offering insights into how these methods address the challenges posed by imprecision and uncertainty in data aggregation.

Laura DE MIGUEL

Universidad Pública de Navarra
Pamplona, Spain

Title: **Generalizations of aggregation functions**

The fusion, or equivalently, the aggregation of information in its broad sense is one of the main step in almost any data processing system. Its principal objective is to seek for a representative value that allows summarizing information from all the given data. Within the field of fuzzy logic, one of the key concepts for information fusion is the notion of aggregation function.

In recent years, numerous studies have emerged demonstrating that the original properties of aggregation functions can be very restrictive. Indeed, we find various examples of functions that even though they do not satisfy to be aggregation functions; they lead to better results in various applications. In this talk, we will present some of the main generalizations of aggregation functions along with their applications.

Abstracts

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On Properties of Mamdani-Assilian Fuzzy Rules – an Open Problem Posed

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Fuzzy rules [4] and fuzzy inference systems have become the central point of fuzzy modeling since the early beginnings of fuzzy systems. Hence, distinct desirable properties of the rules, their models, and the whole systems are studied. The non-conflictness of rules and/or the preservation of modus ponens seem to be considered the most crucial one(s). However, under the standard setting, such properties are semantically equivalent to the continuity of the modeled dependency.

A natural question arises whether such a requirement is consistent with semantics of fuzzy rules. While the answer is positive in the case of implicative rules [2,1], in the case of the more often used Mamdani-Assilian rules [3], we may consider another viewpoint. This article foreshadows another perspective that could lead to the investigation of a desirable property of the Mamdani-Assilian model that is different from continuity. Although it turns out that the above-mentioned non-conflictness of the rules is not the unavoidable property of the Mamdani-Assilian systems, the tools of its investigation based on systems of fuzzy relational equations remain useful even in our investigation. Finally, we formulate an open problem that has the potential to shed a new light on the fundamental question of how Mamdani-Assilian rules precisely work.

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Fuzzy Inference System based on rule comparison measure

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In order to create inference systems able to deal with imprecise information Zadeh presented the generalized modus ponens [3]. In those systems, if-then rules are employed in which the antecedents and consequences are fuzzy sets. This generalized modus ponens is a theoretical concept that can be solved through different mechanisms like Zadeh's compositional rule of inference [2] or others ([1][4][7][8]).

In this contribution, we first introduce the concept of fuzzy rule comparison measure. This measure can be constructed using different indices and measures between fuzzy sets. For each pair of antecedents and consequences of the two rules to be compared, a comparison is done with the selected indices and finally all of this partial results are aggregated into one comparison measure. Lastly, on the basis of this fuzzy rule comparison, we develop a new method in order

to solve the generalized modus ponens via a decision making system with the results of several inference methods. This system is based on the comparison of the fuzzy rules generated by those methods and the original rule base.

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Generalized Takagi Sugeno Kang fuzzy inference system on intensity-modulated radiotherapy treatment selection

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Intensity-modulated radiation therapy (IMRT) is a widely used treatment to combat localized tumors [3]. However, one of its main problems lies in the cost in time and the large number of professional staff required to plan the treatment. As a result, treatment plans are usually planned at once and remain static for the entire course of treatment, which can last for several weeks. This approach is problematic, as a patient's morphology changes daily.

In recent years, in parallel with advances in artificial intelligence, knowledge-based radiotherapy treatment planning tools have been developed with great success [2]. Although these tools can rapidly generate multiple treatment plans, the ultimate decision on optimal treatment is still determined by medical physicists and oncologists. The assessment of treatment quality is linked to solving a linear programming problem with multiple constraints involving doses received by specific risk organs and tumor tissues.

In this research, we propose an explainable decision-making system for determining the quality of IMRT treatments. To achieve this, we address certain limitations of the Takagi-Sugeno-Kang (TSK) [1] model and present a generalized model suitable for problems characterized by substantial uncertainty. This system aims to improve treatment planning by providing a transparent and explainable framework for medical experts, facilitating more informed decision making in adaptive IMRT planning.

Acknowledgement This work was supported by research project PID2022-136627NB-I00.

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A Comparison Between FIS and ANFIS on Chaotic Time Series Forecasting for Forestry Management

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Many industrial sectors rely on time series prediction, including automation, forestry, weather forecasting, medicine, and many more. It happens when any scientific predictions are made based on time stamped evidence from the past. A well-known tool for time series forecasting is the fuzzy inference system (FIS). It was chosen by several academics since it has been proved to deliver reliable forecasting. Adaptive networks, on the other hand, have been integrated into the FIS for time series forecasting. The primary purpose of this research is to compare results obtained with and without the adaptive network. The Takagi-Sugeno-Kang (TSK) FIS model underpins both methods, with type-2 fuzzy sets utilised to express the antecedent and consequent components of fuzzy rules. They are also generated using the Mackey-Glass (MG) time-delay differential equation. The example data utilised for both designs is weather forecasting for forestry management. It is crucial because successful forecasting is dependent on clean, time-stamped data in order to detect actual trends and patterns in historical data. The RMSE calculation yields the best method. The lowest RMSE score gives more accuracy and predicted results when compared to others.

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Modification of Residual LSTM using a new approach to self-attention based on overlap and grouping functions

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Most natural language processing models nowadays rely on the so call attention mechanisms [3]. These mechanisms extract information base on the importance of the given data, hence giving the models the ability to extract more accurate characteristics.

On the other hand, sequential data is usually the field of expertise in which Recurrent Neural Networks shine. Different variations such as LSTM and GRU can be extensively found in the literature.

Given the advantages of both methods, it is natural to combine them when it comes to

solving problems where the data is not only based on natural language but also sequential. This is the case in text translation, where it is necessary to consider the importance of a given word in its context, yet to keep the phrase sequentiality.

In this work, we introduce a new modification of the residual LSTM [1] using a new self-attention method, which aims to extract information from the most important word inside a context. This modification allows us to include both past and contextual data into the residual network.

In order to do so, we propose a new method to aggregate the information extracted by the self-attention mechanism using both overlap and grouping functions [2], thus obtaining new representations of the data obtained from the most important words.

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The Extended TOPSIS Method for Dial-a-Ride Problem with Time Window Using Interval-Valued Intuitionistic Fuzzy Sets

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A The Dial-a-ride problem with time window specifies transportation requests between pairs of origins and destinations within a specific time window. The transportation is provided by a fleet of vehicles based at a common depot. Ultimately, the goal is to find the vehicle route with the least cost that is capable of accommodating all travellers' requests. There are some constraints must be taken in consideration; the vehicle's capacity, route duration and the maximum accepted delay time by travellers. TOPSIS (The technique for ordered preference by similarity to the ideal solution) is one of the well-known methods for multiple attribute decision making (MADM). In this paper, we extend the TOPSIS method within an interval-valued intuitionistic fuzzy (IVIFS) environment and treat the problem as a multiple attribute group decision making (MAGDM), by which all the preferences provided by the passengers (decision-makers) are presented in the form of interval-valued intuitionistic fuzzy decision matrices. We established an optimization model to deter-mine the weights of the attributes and then constructed the weighted collective interval-valued intuitionistic fuzzy decision matrix, to determine both the interval-valued intuitionistic positive-ideal solution and the interval-valued intuitionistic negative-ideal solution. Finally, based on Park et al. distance between IVIFS, the relative closeness of each alternative from the interval-valued intuitionistic positive-ideal solution was ranked. Leading to the collective most desirable alternative(s) selection

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Credibility Theory in Statistical Reasoning with Imprecise Data

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The analysis of imprecise experimental data requires the use of methods that consider two types of uncertainty: randomness and a lack of precision. While classical mathematical statistics copes well with the first type of uncertainty, fuzzy set theory is used to model imprecision. Unfortunately, creating new, hybrid analytical tools by simply adapting known statistical techniques is rarely successful. For example, the absence of suitable models for the distribution of random fuzzy numbers prompts to application of nonparametric approaches. However, fuzzy numbers are not linearly ordered, which makes it impossible to transfer rank tests to the fuzzy domain directly. Therefore, the presence of imprecise data creates the need to develop new statistical inference tools, in particular, statistical tests.

In this presentation, we focus on two-sample tests for fuzzy observations, using the credibility theory by Liu [4] in combination with certain functional data analysis methods. Besides constructing new tests and examining their statistical properties, we want to compare their effectiveness with other previously proposed tests for imprecise data [3,2,1].

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Rough PDA, Rough CFG and its Application to Healthcare System

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The present work proposes a new computational model, Rough Pushdown Automata (R-PDA), which is an extension of Pushdown Automata [3] having amalgamation of Rough Set Theory [2]. The proposed R-PDA model can deal with imprecise and uncertain data. Our work demonstrates the effectiveness of R-PDA in decision making under uncertain conditions. Alongside the R-PDA we further developed the concept of rough context free grammar (R-CFG), and proved its equivalence to R-PDA, i.e., for every R-PDA there is an R-CFG which produces the same language that is accepted by the R-PDA, and vice versa. Some closure properties of R-CFG are also established to demonstrate the efficacy of the proposed model mathematically.
The proposed R-PDA model has potential applications in various domains where uncertainty and incompleteness are common, and it can thereby contribute to the development of more accurate and efficient decision support systems. The research provides insights into the use of rough set theory in automata theory and its potential strength in solving real-world problems. In this era of technological advancement coping with uncertainty is more crucial than ever. Medical domain is no exception. The proposed R-PDA model integrates intuition and logic of a medical practitioner to help in decision making under uncertain conditions of the patient and surrounding ambience. Efficacy of the proposed R-PDA model is demonstrated through a case study of treatment of Asthma, one of the most common chronic diseases in the world.
Although Rough FSA [1] has already been proposed in literature, the concept of R-PDA appears to be novel. Moreover, application of Automata theory is rarely discussed in healthcare system. The proposed model helps in bridging the gap in an effective way.

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Constructions of aggregation functions based on generalized Möbius transform

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Aggregation functions based on fuzzy measures are widely used for data aggregation in various fields where the additivity of probability measures is too restrictive and need to be relaxed by requirement of monotonicity. These fields include decision making, artificial intelligence, economics, and more. In practical scenarios, identification of a fuzzy measure that models interactions between criteria, can be a challenging task. Therefore, a representation of fuzzy measures in terms of some other set functions can be useful. The additive representation of a fuzzy measure is given by the well-known Möbius transform. In fact, the Möbius transform can be defined for any function defined on some finite poset with values in some (semi)ring. Considering so-called pseudo-arithmetical operations on real numbers [3], the generalized Möbius transform can be defined [2, 1].

One of the most widely used fuzzy integrals, the Choquet integral, can be expressed in terms of the Möbius transform. This expression, known as the Lovász extension, has been generalized in various ways [6, 5]. Recently, another approach to constructing aggregation functions in terms of Möbius transform appeared in [4]. In their work, the authors there introduced a new binary operation for real functions on posets called the Möbius product, and proposed a construction of aggregation function based on this operation. This construction, in particular settings, leads to the Lovász extension formula, thus can also be seen as a generalization of that.

In our contribution, we focus on the Möbius product based on generalized Möbius transforms such as possibilistic and pseudo-addition-based Möbius transforms. We also investigate the Möbius product yielding aggregation functions when considering k-additive, k-maxitive and k-pseudo-additive capacities.

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On aggregation of multi-valued data

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As several research results on aggregation functions and their possible applications show, there are situations when it is not enough to work with dimensionless (numerical) inputs and the multi-dimensionality of the aggregated data needs to

be taken into account. Examples include the representation of color images using (R, G, B) triples of pixels intensities, as well as the problem of multi-criteria multi-expert decision making, where several experts rank their preferences for one alternative over another.

Referring to the results obtained in [3], this contribution goal is to convey the method which enables to carry out a fusion process of multidimensional data.

It is based on looking for some sort of an equilibrium (the equation solution) described in terms of Daróczy means ([2]). Since, in general, such means are not monotone (which is one of requirements to be an aggregation function) there was necessary to make some improvements/modifications/generalizations. Those were done in [1] via inf-sup-middle-approach, namely the representative lies in the middle of

$$\sup\{y \mid \sum_{\text{throughout all inputs}} D(\text{input}, y) < 0\} \quad \text{and} \quad \inf\{y \mid \sum_{\text{throughout all inputs}} D(\text{input}, y) > 0\},$$

while D measures deviations.

The method is built up on appropriate multidimensional deviation functions, in order to replace related multidimensional inputs (data sets matrices) by a single multidimensional representative. Moreover, the fact that some real situations are operated by real-valued data instead of values restricted to the unit interval was taken into account.

In order to show the usefulness of this approach, the algorithm will be illustrated on multi-expert decision making problem.

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Some generalizations of the Choquet integral – applied and theoretical

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After the pioneering work of Gustave Choquet [4], based on which the concept of the Choquet integral was built, is the Choquet integral still being studied and generalized by many researchers. Several of the more recent generalizations of the Choquet integral were inspired by the work of Do and Thiele [3] about outer measure space theory. One of these generalizations was proposed by Boczek et. al. in [6] where the authors replace outer essential supremum in Do's and Thiele's construction with a more general conditional aggregation operators. In [1], Pap introduced his modification of work [3] based on sublinear means. He call it the (MC)-integral. Later, Boczek and Kaluszka in [5] introduced an operator, that covers several concepts like mentioned (MC)–integral, two-fold integral [2], the standard Choquet integral, and others.

In our contribution, we shall discuss the mentioned generalizations and introduce a new operator

$$\mathrm{C}^{\mu}_{\mathscr{A},\pi,\sigma}(f) = \sum_{i=1}^{n} \mathrm{CAg}_{\sigma(i)}(f_{\pi}) \left(\mu(E_{\sigma(i)}) - \mu(E_{\sigma(i+1)})\right),$$

where π, σ are permutations of basic set, $\mathrm{CAg}_i(f) = \acute{\mathrm{L}}maths f A^i(f_{\pi}|B) \in \mathscr{A}_i$ is a sequence of conditional aggregation operators, f is an aggregated function, $f_{\pi} = (f(\pi(1)), \ldots, f(\pi(n)))$, and μ is a monotone

measure. This operator connects the idea of integration with respect to conditional aggregation operators with the operator introduced by Boczek and Kaluszka, which is covered by it. This operator covers also some other operators or integrals generalizing the standard Choquet integral. The Choquet integral and its generalizations are widely studied not only from a theoretical point of view but also because of their applicability in many areas. As part of our contribution, we shall present some applications of these generalizations in problems of image processing.

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An extension of the Choquet integral for multi-valued data

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We propose a new generalization of the classical Choquet integral that allows the aggregation of n functions, not necessarily defined on a finite set, resulting in a single scalar output [4]. We study its fundamental properties including subadditivity, comonotonic additivity and convergence theorems, and show that in the case where integrands are defined on a finite set, the integral becomes an aggregation operator of multiple possibly interrelated data vectors. We provide an adaptation of the Sugeno [3] method that permits to compute the new integral of monotone functions when the monotone measure is a distorted Lebesgue measure. Furthermore, we derive several bounds on the integral and show how it can be used to evaluate monotone measures of certain sets.

One of the application inspirations for the construction of the new integral were the works of Sugeno [2] and Yager [1], in which the counterparts of variance and covariance for the integrals with respect to non-additive measures are presented. Such numerical characteristics indicate how strong the relationship is between two functions (e.g. fuzzy sets), given a monotone measure that assigns different degrees of importance to the arguments of these functions in a given decision problem. Characteristics of this type, introduced on the basis of the proposed integral, can be used to extend classical decision-making methods, originally designed for probabilistic uncertainty, to account for measure-based uncertainty. Another context where the integral can be utilized is in the definition of the concept of vector data averaging.

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Sugeno Integral: compatibility and its relation to distributivity

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Sugeno integral is defined as an idempotent weighted lattice polynomial in the framework of bounded distributive lattices [1]. There are several axiomatic characterizations of this integral. One of them characterizes Sugeno integral as a compatible aggregation function uniquely extending a given L-valued capacity [4]. The mentioned property of unique extension will be discussed and its relation to the distributivity of the underlying bounded lattice L will also be addressed [3]. Among others, we have shown that these two notions are equivalent. As a byproduct, an alternative proof of Iseki's result [2], stating that a lattice having a prime ideal separation property for every pair of distinct elements is distributive, is provided.

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Fuzzy Equivalence Relations as Similarity Measurement in Hierarchical Clustering Algorithms

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In clustering process, one of important measures for grouping data in different clusters, are the similarity and dissimilarity of two objects. If dissimilarity can be obtained by using Euclidean, Manhattan, Hamming or other distances, similarity of two objects are inversely proportional to dissimilarity and can be expressed via distance between two objects in different ways:

$$s(x,y) = e^{-d(x,y)}, \quad s(x,y) = \frac{1}{1+d(x,y)}.$$

Such similarity measurement is a special cases of fuzzy equivalence relations, namely fuzzy equivalence relations, where the transitivity is fulfilled for the product and Hamacher t-norm. Note, that there exists an important tool for the construction and study fuzzy equivalence relations (T-equivalences) involving only one-argument real function (additive generator) and metric. In work [1] was shown, how fuzzy equivalence relations and aggregation of corresponding equivalence relations was involved in clustering process. In this work is considered hierarchical clustering, and as similarity measurement of objects and clusters is considered fuzzy equivalence relations [2] for different t-norms:

- Lukasiewicz: $E_L(x,y) = \max(1 - d(x,y), 0)$;
- Product: $E_P(x,y) = e^{-d(x,y)}$;
- Hamacher: $E_H(x,y) = \frac{1}{1+d(x,y)}$.

With such similarity measurement, based on fuzzy equivalence relations, can be obtained clustering performance metrics - potential function, that shows clusters are good separated or not. In the work is provided comparative analysis of clustering results with different t-norms of fuzzy equivalence relations. In presentation will be shown advantages of using fuzzy equivalence relations as similarity measurement, over others standart similarity measures.

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Fuzzy Distance-based Inequality Relations and Equivalences in Clustering Problems

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Clustering is usually thought of as the problem of establishing groups in a set (say $A = \{a_1, ..., a_n\}$) of objects so that the same group (called a cluster) contains objects that are more similar (in some sense) to each other than to objects in other groups (clusters) [2]. If objects are embedded in the Euclidean space $\mathbb{R}^m$, then a distance-like function $d : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^+$, can be used to measure the similarity/dissimilarity of objects, where the distance function d is any function that satisfies the following properties: (i) $d(x,y) = 0$ if and only if $x = y$; (ii) $d(x,y) = d(y,x)$; and (iii) for every fixed y, $d(x,y)$ tends to infinity if the Euclidean norm of x tends to infinity.

In our work we propose to construct fuzzy inequality relations D (or generalized distances [1]) based on the distance-like function d as follows: $D(x,y) = s^{(-1)}(d(x,y)/t)$, where s is an additive generator of an (Archimedean) t-conorm S, and $s^{(-1)}$ is its pseudo-inverse. Using D, we find the centers of clusters as points in $\mathbb{R}^m$ that give minimum to the function $\sum_{i=1}^n D(a_i, x)$, i.e. we characterize these centers as follows: $\arg\min_{x\in\mathbb{R}^m} \sum_{i=1}^n D(a_i, x)$. For non-linear functions s we expect to obtain the local mass centers. The parameter t is responsible for having reasonable number of clusters. Note, that if d is a metric, then D is a fuzzy inequality relation. It also should be mentioned that solving the problem: $\arg\min_{x\in\mathbb{R}^m} \sum_{i=1}^n d(a_i, x)$, we obtain the mass center of all objects in A, rather than local mass centers. The dual problem is formulated as follows: $\arg\max_{x\in\mathbb{R}^m} \sum_{i=1}^n E(a_i, x)$, where $E(x,y) = f^{(-1)}(d(x,y)/t)$, f is an additive generator for a t-norm T and $f^{(-1)}$ is its pseudo-inverse. In this case E is a fuzzy equivalence relation. We expect that using fuzzy distance-based inequality relations and equivalences help us to pose the problem analytically, and also select the most suitable method for solving it.

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Fuzzy Equivalence Based Numerical Algorithm for Solving Multi-objective Linear Programming Problems

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When dealing with multi-objective linear programming (MOLP) problems, a commonly employed strategy is to aggregate individual objective functions and optimize the resulting aggregation function [2], [1]. One of the approaches, studied in [3], utilizes fuzzy orderings and transform the MOLP problem into the form $\max_{y\in D} \min_{x\in D} P(x,y)$, where P is the fuzzy ordering and D denotes the search space.

It is worth noting that the choice of T-norm for constructing fuzzy orderings has been found to be immaterial, and thus, the Lukasiewicz T-norm is typically preferred due to its simplicity. However, the function $f(y) = \min_{x\in D} P(x,y)$ is piecewise smooth, that poses a challenge, particularly for steepest descent algorithms.

Recognizing that, for a fixed y value within the set D, selecting x exclusively from the vertices of D suffices, we present a novel methodology. This methodology is based on the construction of a linear combination of subgradients, with weights computed using fuzzy equivalence relations. The proposed approach, which replaces the conventional steepest descent method, seeks to address the limitations imposed by the piecewise smooth nature of the problem.

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Adaptive proximity relation and conformance measure in data evaluation

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Generally, an item (product, service and the like) is expressed as a vector of attributes (e.g., colour, price, opinion, pollution, presence of a specific feature). Evaluating the most suitable ones should closely follow the raised requirements [5]. Aggregation functions calculate the overall satisfaction degrees. Before this

step, matching degrees to each atomic condition should be assigned. It is not an easy task when data are categorical (ordinal, nominal and binary as a special case) and non-precise numerical data. Moreover, the same attribute can be expressed as categorical or numerical, i.e., value 150 and term *rather small* for different entities. The proximity relation proposed in [4] is an option to express proximity between requirements and data. In this work, we discuss a proximity relation capable of handling similarities between categorical values (including discretized data) by linguistic terms expressed via triangular fuzzy numbers. To reach this goal, we should keep the matrix of proximity values consistent and the family of linguistic proximities should meet the quality measures proposed in [3]. The next step is calculating conformance between requirements and data. To meet it, the conformance measure proposed in [4] and modified in [5] is extended to cover fuzzy numbers. Next, we propose a convex combination of possibility and necessity measures to cover the smooth translation from the optimistic to the pessimistic evaluation. The results are demonstrated in an illustrative example.

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Retinal Vessel Segmentation based on Fuzzy Histogram Feature Enhancement with the Improved Bonferroni mean-based Pre-aggregation Operators

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Retinal vessel segmentation plays a crucial role in enhancing diagnostic ability in ophthalmology. However, manual segmentation of vessels is a tedious task. Thus, the investigation of automated retinal vascular feature segmentation has emerged as a substantial area of research in recent times [8] and broadly categorized into handcrafted filter based methods [7, 6, 5] and deep learning methods [4, 3, 2]. Early stages filter-based methods are unable to extract dissimilar types of vessel features. On the other hand, the existing deep learning methods have enormous computational complexity and are data-dependent for training purposes.

In order to overcome the mentioned limitations, the aim of this work is to propose a novel vessel segmentation approach that consists of two components: (a) retinal vessel extraction by fusing color channel information with the help of interrelationship handling improved Bonferroni mean-based pre-aggregation operators (IBMPO), and (b) enhancement of vessel appearing in the low contrast region through fuzzy histogram technique based on the prior feature intensity information evaluated through IBMPO. Initially, the input image is processed to extract a prominent vascular feature map by eliminating the background artifacts from each color space, say I_1, I_2, and I_3 followed by the magnitude intensity evaluation of the vessel features by fusing I_1, I_2, and I_3 with the suitable form of IBMPO. The intensity $T(i,j)$ in the feature map T is computed as given below.

$$T(i,j) = \frac{1}{56h}\sum_{i=1}^{8}\sum_{j=1,j\neq i}^{8} h\Big(\frac{u_i^2+u_j^2}{u_i+u_j}\Big) \quad (1)$$

where, $u_k = |r'_{(i,j)} - r'_{A(k)}| + |g'_{(i,j)} - g'_{A(k)}| + |b'_{(i,j)} - b'_{A(k)}|$, $k \in \{1,2,...,8\}$, $A(k) \in \{(i-1,j-1),...,(i+1,j+1)\}$ 8-point neighbourhood, $r'_{(i,j)}$, $g'_{(i,j)}$, and $b'_{(i,j)}$, denote the gray levels in I_1, I_2, I_3, respectively.

The advantage of using IBMPO is that it considers the interaction between the vessel pixels and their neighborhood in the direction of increasingness in such a way that it preserves the geometrical characteristics of vessels. Further, the extracted vessels from the initial stages are enhanced by constructing a fuzzy histogram using the prior vessel feature intensity of T employing Eq. 1. The fuzzy histogram is evaluated by constructing the following membership function:

$$\nu_{T(x,y)l} = max\Big(0, 1 - \frac{|T(x,y)-l|}{4}\Big) \quad (2)$$

where, $l \in \{0,1,...,L-1\}$ and L is the number of gray levels.

The experiments are accomplished over two publicly available datasets, DRIVE [1] and STARE [7], which illustrate the significance of the proposed approach compared to existing methods with fast segmentation.

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Constructing uninorms on bounded lattices via uninorms

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Uninorms on the real unit interval $[0, 1]$, introduced by Yager and Rybalov [2], are a special kind of aggregation operators generalizing the concepts of triangular norms (t-norms, for short) and triangular conorms (t-conorms, for short) and allowing for their neutral element to lie anywhere in $[0, 1]$. Afterward, Karaçal and Mesiar [5] extended the concept of a uninorm from the unit interval to bounded lattices. They also demonstrated the presence of a uninorm with the fixed neutral element on a bounded lattice. Since then, uninorms on bounded lattices have been studied extensively in a manner similar to their investigations on the unit interval. In particular, a lot of construction approaches for uninorms on bounded lattices have been presented in the literature, including the ones by using t-norms (or t-conorms) [10,9,8], closure (or interior) operators [4], t-subnorms (or t-superconorms) [6,1], and additive generators [7]. However, these construction methods mainly focus on the presence of t-norms and t-conorms.

In recent years, Xiu and Zheng [3] have brought a new perspective to discuss the constructions of uninorms on a bounded lattice L by extending a uninorm defined on a sublattice of L to a uninorm on L. Notice that their approaches generalize some existing methods to obtain uninorms on bounded lattices (e.g., [8,5]).

In this contribution, we propose new approaches for generating uninorms on a bounded lattice L by considering the existence of a uninorm instead of a t-norm (t-conorm) on a sublattice of L. More precisely, we extend the methods to obtain uninorms via t-norms (t-conorms) to those via uninorms defined on a sublattice of L. Since uninorms are more general than t-norms and t-conorms, our methods based on the existence of a uninorm are more effective than some methods in the literature generated by t-norms and t-conorms. It is worth noting that some known construction approaches for uninorms (t-norms or t-conorms) on bounded lattices can be derived from our tools.

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Approximations using triangular norms

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In applications of fuzzy systems, the choice of triangular norms (representing conjunctions in fuzzy logic) is one of the questions requiring attention [3]. Unless we restrict to a specific class of triangular norms, we have a huge range of possibilities. One approach is to collect data in the form of desired values in specific points of the domain. If these values are taken as precise, they lead to a system of equations hard to solve and even the existence of a solution is difficult to decide.

It is often the case that the data are not precise and exact fit to them is not required. Then we look for an approximation of the data by a general triangular norm. Associativity is the property that can hardly be involved in the approximation. The usual trick is to represent the triangular norm by its generator (additive or multiplicative). This is a unary function describing the triangular norm in such a way that associativity is ensured. New problems arise with the use of generators:

1. They are not uniquely determined.
2. They must be monotonic, thus standard (linear) approximation methods cannot be applied.
3. The shape of the generator is not directly related to that of the triangular norm [5,4,2].

Gleb Beliakov [6] suggested a solution which finds a monotonic generator by a least square approximation with a spline. However, he optimized the residual errors in the range of additive generators, not in the original range of the triangular norms. We show [1] that such solutions may be far from the optimum of the original task (minimization of the sum of squares of differences between the given point and the values of the triangular norm). We propose a direct solution of the least squares problem. Although computationally more demanding, it is feasible, and it leads to nearly optimal solutions of the original problem.

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The (maximal) chain as a tool for generalization of the discrete Choquet integral

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The discrete Choquet integral on nonnegative reals is based on a distinguished permutation making the inputs reordering nonincreasingly or nondecreasingly. This rearrangement, or permutation respectively, relates to a maximal chain $\mathscr{C} \in \mathcal{C}_n$ on the power set $2^{[n]}$. This observation leads us to introduce a new class of discrete functionals called *the maximal chain-based Choquet-like integrals* (shortly, MCC-integrals), see also [3]. More precisely, the MCC-integral w.r.t. $\mathscr{C} \in \mathcal{C}_n$ and $\otimes : (-\infty, \infty)^2 \to (-\infty, \infty)$ is defined by

$$\mathrm{MaxCh}^{\otimes}_{\mathscr{C}}(\mathbf{x}, \mu) = \sum_{i=1}^{n} x_{\sigma_{\mathscr{C}}(i)} \otimes (\mu(C_i) - \mu(C_{i-1})), \tag{1}$$

where $\mathbf{x}_{\sigma_{\mathscr{C}}}$ denotes the rearranged vector of $\mathbf{x} : [0, \infty)^n \to [0, \infty)$ by the permutation $\sigma_{\mathscr{C}}$ corresponding to $\mathscr{C} \in \mathcal{C}_n$, and $\mu : 2^{[n]} \to (-\infty, \infty)$ is a game. In addition to discussing basic properties of (1) with respect to properties of $\otimes$, we will provide various representations as well as a symmetric and asymmetric extension of (1) for real-valued inputs giving new insight into the construction of fuzzy integral quadruplets [1]. A relation with results of [5] is given. We will show that the related concept

$$\mathrm{CC}^{\otimes}_{\mathscr{C},\mathcal{A}}(\mathbf{x}, \mu) = \sum_{i=1}^{m} \mathsf{A}(\mathbf{x}|C_i \backslash C_{i-1}) \otimes (\mu(C_i) - \mu(C_{i-1}))$$

studied in [2], which is based on an arbitrary (not only maximal) chain $\mathscr{C} = \{C_0, C_1, \ldots, C_m\}$ with $m \in [n]$ and FCA $\mathcal{A}$ introduced in [4], enables to cover some scientometric indices and provides a good motivation for further research in this framework.

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On conditional OWA-operators

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In the multitude of existing aggregation techniques, ordered weighted averaging (OWA) operator introduced in [1] is one of the most widely used. The OWA-operator is a symmetric aggregation operator that combines both conjunctive and disjunctive behaviors. This concept has aroused a great deal of interest in the scientific field, and a considerable number of publications have been produced studying its properties, extensions, and possible applications.

An important step towards a deeper understanding of the behavior of OWA-operators and their incorporation into the broader landscape of aggregation functions was the work of Murofushi and Sugeno [2]. In their research, the authors demonstrated that the Choquet integral can represent several useful quantities, such as the supremum, infimum, essential supremum, essential infimum, median, α-quantile, and L-estimator.

In this contribution, we focus on the currently studied generalizations of Choquet integrals using conditional aggregation operators [3]. We have developed the appropriate calculus for the generalized Choquet integral with respect to the generalized level measure [4] on a discrete universe. Since the Choquet integral with respect to a symmetric measure is an OWA-operator, the generalized Choquet integral with respect to the symmetric measure can be regarded as a new type of OWA-operator. We introduce several extensions of OWA-operators based on conditional aggregation and investigate their properties. Furthermore, we aim to demonstrate the practical value of the suggested extension throughout real-world application.

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Characterizing a Semigroup by its Fuzzy Interior Antiideals

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Fuzzy sets serve as an extension of traditional sets, capable of handling imprecise information. On the other hand, semigroups represent a broader category than groups, finding utility across various interdisciplinary domains. Diverse subsets of semigroups have been investigated to enhance comprehension of their practical applications. Given the significance of both these ideas, exploring fuzzy subsets within semigroups carries substantial importance. In this paper, we characterize a semigroup through one of its (fuzzy) subsets; (fuzzy) interior antiideals. More precisely, we define interior antiideals of a semigroup, illustrate it by examples, and study its properties. Furthermore, we extend this notion into a fuzzy context, examining the characteristics of fuzzy interior antiideals of a semigroup.

Keywords and phrases: Antideal, interior antiideal, fuzzy interior antiideal, level set.

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A dagger kernel category of complete orthomodular lattices

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Dagger kernel categories have been introduced in [4] as a simple setting in which one can study categorical quantum logic. The present paper continues the study of dagger kernel categories in relation to complete orthomodular lattices in the spirit of [3].

Definition 1 The category **SupOMLatLin** has complete orthomodular lattices as objects. A morphism $f\colon X \to Y$ in **SupOMLatLin** is a function $f\colon X \to Y$ between the underlying sets such that there is a function $h\colon Y \to X$ and, for any $x \in X$ and $y \in Y$,

$$f(x) \perp y \text{ if and only if } x \perp h(y).$$

We say that h is an *adjoint* of a *linear map* f. It is clear that adjointness is a symmetric property: if a map f possesses an adjoint h, then f is also an adjoint of h.

Moreover, a map $f\colon X \to X$ is called *self-adjoint* if f is an adjoint of itself.

The identity morphism on X is the self-adjoint identity map $\mathrm{id}\colon X \to X$. Composition of $X \xrightarrow{f} Y \xrightarrow{g} Z$ is given by usual composition of maps.

In particular, we show that the category of complete orthomodular lattices

SupOMLatLin is a dagger kernel category and an involutive quantaloid. We describe dagger biproducts and free objects in **SupOMLatLin**. We show that the endomorphisms on a complete orthomodular lattice form an involutive quantale (see [2]) that is simple. Hence, we obtain that complete orthomodular lattices are modules over an involutive quantale.

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Some Remarks on Categories of Fuzzy Morphological Spaces

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Mathematical morphology arose in the 70[ies] of the last century in the works of G. Matheron and J. Serra (see, for example, [2]) and was initially driven by the needs of practical geology. However, it soon found important applications in other fields, particularly in image processing, and is now widely studied and applied by many researchers. The first version of mathematical morphology in the context of fuzzy sets was presented in the paper by De Baets et al. [5]. As in the case of "classical" mathematical morphology, the basis here is the linear structure of an Euclidean space $\mathbb{R}^n$ and a chosen subset $B \subset \mathbb{R}^n$, intuitively small and called the structuring element. Again, as in the "classical" mathematical morphology, the linear structure and the structuring element give rise to (in this case fuzzy) basic operators of mathematical morphology: erosion $\mathscr{E}$ and dilation $\mathscr{D}$ connected by the Galois connection.

When working with (fuzzy) morphological spaces, a researcher sometimes has to make various modifications of the original morphological structure, in particular, to compress it, to reduce the number of dimensions involved in the structure, to combine several structures into a new one, and others. In all these cases a question appears (at least implicitly) what mappings are allowed during such transformations. In turn, in order to correctly define the requirements for these mappings, the problem of defining the category of (fuzzy) morphological spaces naturally arises. The main purpose of our presentation is to offer a categorical view of fuzzy mathematical morphology in the spirit of the article [5]. Particular, in this category we emphasize the role of the structural element, which together with the linear structure of the space, is crucial for most applications.

In conclusion, it should be noted that there are abstract, algebraic approaches to the subject of mathematical morphology based on the properties of erosion and dilation operators connected by the Galois connection. In turn these approaches allowed to develop a categorical view of mathematical morphology, see, e.g. [4] et al. However, the disadvantage of the abstract approaches is that the role of the structural element is practically lost and therefore their use for the practical applications of mathematical morphology seems to be very limited.

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Counting fuzzy subgroups in U_{6n}

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In this talk we review existing methods for counting the number of all (normal) fuzzy subgroups of some group and present an algorithm for the case of finitely presented groups $U_{6n} = \langle a, b \,|\, a^{2n} = b^3 = 1, bab = a\rangle$, $n \in \mathbb{N}$. For cyclic groups $\mathbb{Z}_n$ there is a closed form formula for counting (depending on a canonical prime factorization) ([2]), we present partial results for certain forms of n in the case of U_{6n} extending results from [6].

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The Global Pension Index in the Context of Multicriteria Decision Making

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The Global Pension Index serves as a critical benchmark for assessing the retirement systems of nations worldwide, [2]. This study explores the optimization of the Global Pension Index through the application of multicriteria

decision-making methods, [4]. Specifically, we compare the efficacy of two prominent methodologies, Saaty's Analytic Hierarchy Process and the Utility Function Method, in evaluating subindices and assigning weights to key indicators, [1, 3].

By employing these methodologies, we aim to enhance the accuracy and relevance of the index, considering the dynamic nature of global pension systems. Moreover, we investigate the statistical properties of the weights obtained using each method across different levels of uncertainty, shedding light on the robustness of the solutions provided. Our findings offer insights into the best solution for evaluating and ranking pension systems, addressing the sensitivity of the index to changes in the chosen evaluation method. Additionally, we examine the implications of altering the weights on the Global Pension Index, demonstrating the potential impact on the overall assessment of retirement systems.

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Optimizing Self-Attention through Aggregation Techniques

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In the realm of modern artificial intelligence, Transformers have earned their place as a groundbreaking architecture for various natural language processing and machine learning tasks [1]. A central feature of Transformers is their self-attention mechanism, a key component that allows the model to weigh the significance of different input elements dynamically. This dynamic attention mechanism has been a game-changer in tasks like machine translation, text generation, and image classification. However, Transformers typically employ a single aggregation method, known as the product sum, within their self-attention mechanism.

The art of aggregation shines when handling input data with intricate interactions. It has found its application in various domains, including decision-making, brain-computer interfaces, and classification tasks [3, 4]. Moreover, a number of innovative generalizations have emerged, often surpassing the performance of traditional aggregation methods. Notable examples include the CF_1F_2-Choquet [3], and the Vector Choquet integrals (VCI) [2].

Our primary goal in this study is to enhance Transformer performance by incorporating advanced aggregation strategies into the self-attention mechanism. We anticipate that this

fusion will lead to improved efficiency and effectiveness.

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A fuzzy algorithm for image rescaling and its comparison with clasical methods of digital image processing

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Images are an indispensable tool in numerous fields of research and they have also achieved a concrete impact on daily life. The acquisition of a digital image from a given device is a physical process that allows to obtain a two or three dimensional signal; the captured data must be converted using appropriate algorithms.

During this phase the acquisition tools and the corresponding procedure of reconstruction, allow to obtain an image that is characterized by a natural degree of approximation and therefore of uncertainty. In this situation, among the various approachs available in the literature there is also the possibility to apply fuzzy algorithm for digital image processing, as done for example in [8,1,3]. On the other end, it is well-known that digital images are often represented as discontinuous signals and treated resorting to sampling-type algorithms.

The aim of this talk is to present a comparison among the algorithm for image rescaling introduced in [3] with some other existing alghoritms such as the so-called sampling Kantorovich (SK) algorithm [4,5] and the classical bilinear and bicubic algorithms. Note that, the SK algorithm is a recent tool for image rescaling and enhancement that revealed to be usefull in several applications to real world problems [6,2].

The comparison among the above mentioned algorithms (all implemented by MatLab programming language) will be done in term of suitable indexes such as the Peak-Signal-to-Noise-Ratio (PSNR) and the likelihood index S introduced in [7]. Moreover, also the CPU time of the considered algorithms are analyzed.

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Aggregation functions and their association with linear splines

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Linearity is an important concept, especially in engineering applications. We consider continuous functions $F : [0,1]^n \to [0,1]$, i.e., continuous fusion functions, that are piecewise linear, i.e., linear on members of a simplices partition $\{S_1, \ldots, S_k\}$, $k \geq n!$ (linear splines). We consider a relaxed form of partitions, where any two different members have an intersection with the zero Lebesgue measure in the simplex case.

Definition 1 Consider a finite simplex partition $\mathbb{S} = \{S_1, \ldots, S_k\}$ of $[0,1]^n$ (i.e., $\lambda_n(S_i \cap S_j) = 0, i \neq j$ for each $i, j = 1, \ldots, k$, where λ_n is the standard Lebesgue measure on boolean subsets of $[0,1]^n$, (i.e., ordinary crisp subsets), and $\bigcup_{i=1}^{k} S_i = [0,1]^n$). The considered piecewise linear function F is assumed to be linear on each simplex S_i, i.e.,

$$F(x_1, \ldots, x_n) = \sum_{j=1}^{n} a_{i,j} x_j + b_i \ \in [0,1]$$

for each $(x_1, \ldots, x_n) \in S_i, i = 1, \ldots, k$.

Linear splines extending boolean functions in dimension $n = 2$ were presented by Mesiar at EUSFLAT 2017 in Warsaw. Motivated by this idea, we present and discuss now linear spline fusion functions. The aim of this contribution is to study and characterise linear splines, which are also aggregation functions. We focus on

- $n = 2$ for typical types of aggregation functions, such as triangular norms, copulas, etc. As typical example one can recall the Lukasiewicz t-norm T_L and ordinal sums with T_L summands.
- n-ary integral based aggregation functions with no condition on n. As typical examples one can recall the Choquet, the Sugeno and the Shilkret integrals.

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On aggregation functions based on admissible orders on the sets of discrete fuzzy numbers and discrete Z-numbers

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The construction of families of aggregation functions on bounded lattices is currently a hot topic. In this context, the introduction of the concept of admissible order [6] has opened new research lines in the study of different classes of aggregation functions on totally ordered sets based on this type of linear orders.

Recently, admissible orders on the set of discrete fuzzy numbers, as well as, on the set of discrete Z-numbers have been proposed in [2,3]. Discrete fuzzy numbers, and especially, those whose support is a closed interval of a finite chain $L_n = \{0, 1, ..., n\}$ (usually denoted by $\mathscr{A}_1^{L_n}$) have been widely investigated in the literature, since they have proved to be useful for the design of several computational linguistic models [5,4] in which experts can perform their evaluations in a very flexible way. Up to the results presented in [2], the algebraic structure on which these linguistic models were based was a bounded lattice, where the partial order was constructed from extensions of the classical min and max functions. In this framework, several types of aggregation functions were studied in [1]. However, now that admissible orders on this set are available, these computational linguistic models may be enriched and with this aim, in this work, for the first time, we introduce a novel construction method of aggregation functions based on admissible orders defined in $\mathscr{A}_1^{L_n} \times Y_m$ (discrete fuzzy numbers on $\mathscr{A}_1^{L_n}$ with a finite set of membership values). Furthermore, we introduce also aggregation functions based on admissible orders for the set of discrete Z-numbers whose components have a finite set of membership values. These construction methods rely on aggregation functions defined on L_n, a framework where important classes of aggregation functions have been defined such as t-norms, uninorms, etc. Finally, it is proved

that the proposed aggregation functions on the sets of discrete fuzzy numbers and discrete Z-numbers fulfil the same additional properties that are satisfied by the underlying aggregation functions on L_n.

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Some kinds of threshold generated fuzzy implications

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Fuzzy implications are one of the main operations in fuzzy logic. This is why new families of these connectives and their properties are the subject of study ([4]).

One of the directions of such research is considering an ordinal sum of fuzzy implications on the pattern of the ordinal sum of t-norms.

In this contribution, some ways of generating fuzzy implications from given ones are presented. And example of such constructed family is an e-threshold generated implication [3]. This method was modified later by Z.-H. Yi and F. Qin [1] to more components and was called the extended threshold generation method. Another possibility is a vertical e-threshold generated implication [2] proposed by S. Massanet and J. Torrens.

In this contribution, we propose a generalization of these methods that allows to tailor the values of fuzzy implications for specific application. Properties of the fuzzy implication created by the use of this method, depending on the properties of its generators, are examined. Some comparisons to existing methods of construction are presented.

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Close-by-One strategy for computing the fuzzy concept lattice

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In formal concept analysis (FCA), there are two main knowledge structures, namely the the concept lattice and the basis of attribute-implications. In this paper we focus on the former. The Close-by-One (CbO) [2] strategy, mainly based on the set of extents being a closure system, is one of the best known strategies for developing algorithms to compute the concept lattice. The first approach using this strategy is the well-known NextClosure algorithm.

Recently, several improvements of this strategy have been published, among them the one known as FastCbO [1]; and the InClose family of algorithms [3]. However, the contributions in the fuzzy framework are mostly based on scaling the fuzzy context, applying the crisp algorithm to the crisp one and descale the results. Our intuition is that the development of a native fuzzy algorithm avoiding the scaling would speed up the computation. Furthermore, one could take advantage of the perks of the fuzzy setting instead of translating the problem to a crisp one, making the pruning techniques more efficient and therefore avoiding unnecessary computation.

In this paper, we present the extension of CbO-like algorithms to a native fuzzy environment, without scaling, and combining the advantages of the different algorithms to obtain faster results with less computational load. The soundness of these algorithms is presented together with a comparison with existing strategies to show the improvement in both time, number of intents computed and number of tests performed.

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An FCA-based approach to RDF

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Graph-based data representation is gaining popularity. However, as much as it provides a rich form of describing relations between entities and entities themselves, it creates challenges in identifying entities that are matched—ideally or not—via some specific set of relations.

In this work, we investigate building a connection between RDF and FCA. The proposed approach transforms an RDF graph, where vertices represent objects of different types and edges represent relationships between these objects, into a series of bipartite graphs. It is achieved by separating edges representing specific relationships, resulting in a clear representation of the relationship of interest without clutter.

To address this issue, we propose a bond-based construction of rigorous and benevolent compositions of bipartite graphs [3]. These bipartite graphs are extracted from RDF graphs and combined—using the proposed construction—with external information related to the graphs' entities. This information, which somehow represents the degree of similarity between the vertices to be composed, can be better represented using many-valued truth degrees instead of binary values [2]. This is because it allows to capture more subtleties of the relationship of similarity between the entities, as opposed to binary values which only allow for a simple "yes or no" answer. This is particularly relevant when dealing with complex real-world scenarios, where one is likely to find gradual relations between entities.

In future work, we aim to develop new strategies for building compositions that lie somewhere between the two extremes of the rigorous and the benevolent bonds. These approaches may involve the use of intermediate quantifiers [4,1], allowing for a more nuanced representation of the degree of relationship between entities. Such strategies could prove useful in a variety of contexts, such as recommendation systems or data analysis, where a more specific understanding of the relationships between entities is desirable.

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From Kolmogorov Superposition Theorem to the Inverse F-Transform Representation

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The purpose of this contribution is to show the connection and, in a sense, the similarity between the known classical and non-classical results on the representation and approximation of a continuous function.

Let us start with a question: are there genuine continuous multidimensional real functions? Essentially, this question David Hilbert posted at the Second International Congress of Mathematicians, held in Paris in 1900 as one of the 23 problems.

Hilbert's 13th problem conjectured that there are continuous functions of many variables that cannot be expressed as composition and addition of continuous functions of two variables.

It took over 50 years to prove that Hilbert's conjecture is false. In 1956, Kolmogorov proved a remarkable result: any continuous function of any finite number of variables can be expressed through the composition and addition of continuous functions of three (or fewer) variables. In 1957, Arnold showed that 3 could be replaced by 2, thereby proving Hilbert's conjecture false. Shortly thereafter, Kolmogorov showed [3] that 2 can be replaced by 1. The cited below theorem is the reply to the Hilbert's 13th problem.

Theorem 1 (Kolmogorov, Arnold, Kahane, Lorentz, Sprecher [2])
For any $n \in \mathbb{N}$, $n \geq 2$, there exist real numbers $\lambda_1, \ldots, \lambda_n$ and continuous functions $\phi_k : \mathbb{I} \to \mathbb{R}$, $k = 1, \ldots, 2n+1$, where $\mathbb{I} = [0,1]$, with the property that for every continuous function $f : \mathbb{I}^n \to \mathbb{R}$ there exists a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that for each $(x_1, \ldots, x_n) \in \mathbb{I}^n$,

$$f(x_1, \ldots, x_n) = \sum_{k=1}^{2n+1} g(\lambda_1 \phi_k(x_1) + \cdots + \lambda_n \phi_k(x_n)). \tag{1}$$

Our contribution is based on a careful analysis of the proof of the Kolmogorov superposition theorem proposed by the Swedish mathematician Torbjörn Hedberg [4] and repeated in [2]. We aim to show that the functions $\phi_k(x_1), \ldots, \phi_k(x_n)$ discussed above in (1) create a fuzzy partition similar to that considered in [1]. Based on this similarity, we plan to show that the structure of the expression (1) is similar to the inverse F-transform formula.

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On approximation of lattice integral transforms

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In [2], lattice integral transforms were introduced to generalize and consequently extend the approximation abilities of lattice fuzzy transforms proposed by Perfilieva in [1], which can be used in signal and image processing, compression, denoising, data analysis, etc. Without going into details, for a given fuzzy measure space $\langle X, \mathscr{F}, \mu \rangle$, an integral kernel $K : X \times Y \to L$, and $\star \in \{\otimes, \to\}$, the lattice integral transform is defined as the map $F^{\star}_{(K,\mu)} : \mathscr{F}(X) \to \mathscr{F}(Y)$ given by

$$F^{\star}_{(K,\mu)}(f)(y) = \int_X^{\otimes} K(x,y) \star f(x)\, d\mu, \tag{1}$$

where $\int_X^{\otimes}$ is the Sugeno-like integral introduced in [3,4], and $\otimes$ and $\to$ is the multiplication and residuum operation in the complete residuated lattice L, respectively. Thus, a lattice integral transform maps the lattice-valued functions over X to the lattice-valued functions over Y, in a similar way to the standard integral transforms such as the Fourier or Hilbert transforms for real

or complex functions. Lattice fuzzy transforms are then special cases of lattice integral transforms where the smallest and the largest fuzzy measures are considered.

In [1], Perfilieva showed, among other things, that a suitable composition of a direct upper (lower) lattice fuzzy transform and an inverse upper (lower) lattice fuzzy transform approximates the original function from above (below). In terms of lattice integral transforms, the following inequalities were proved:

$$F^{\otimes}_{(K^{-1},\mu_Y^{\top})} \circ F^{\rightarrow}_{(K,\mu_X^{\perp})}(f)(x) \leq f(x) \leq \\ \leq F^{\rightarrow}_{(K^{-1},\mu_Y^{\perp})} \circ F^{\otimes}_{(K,\mu_X^{\top})}(f)(x) \quad (2)$$

for any $f \in \mathscr{F}(X)$ and $x \in X$, where K^{-1} is the inverse integral kernel (i.e., $K^{-1}(y,x) = K(x,y)$ for $x \in X$ and $y \in Y$), and $\mu_X^{\perp}$ and $\mu_X^{\top}$ denote the the smallest and the largest fuzzy measures on X, respectively, and similarly for Y. A natural question is whether the same or similar inequalities hold for lattice integral transforms, i.e. for more general fuzzy measures.

In addition, an interesting and challenging question arises whether we can express the approximation quality of the composition of lattice integral transforms, which means estimating the closeness of the original function and its reconstruction, i.e.,

$$F^{\rightarrow}_{(K^{-1},\nu)} \circ F^{\otimes}_{(K,\mu)}(f)(x) \approx f(x), \quad x \in X,$$

for a suitable setting of the fuzzy measure ν, and similarly for the reverse composition.

The aim of the presentation is to provide answers to the above questions. We introduce the concept of the inverse integral kernel (so-called Q-inverse for a fuzzy relation Q on X) which plays a crucial role in the reconstruction of the original function using the composition of lattice integral transforms. Next, we generalize the inequalities in (2) that concern the upper and lower approximations. Finally, we introduce a modulus of continuity for lattice-valued functions using which we estimate the quality of approximation of the reconstructed function. As a corollary, we get that under certain conditions on integral kernels, the extensional functions with respect to Q can be ideally reconstructed, i.e., the original and reconstructed functions coincide.

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F-transform Utility in the Operational-Matrix Approach to the Nonlinear Volterra Integral Equation

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This research is a smart combination of F-transform and Fixed Point Theorem to approach the nonlinear Volterra integral equation [1]. The general form of non-linear Volterra equation of the second kind has the form

$$y(t) = f(t) + \int_0^t K(s,t)\phi(y(s))ds, \quad (1)$$

where the functions: source $f \in C^1[0,T]$, kernel $K \in C^2([0,T] \times [0,T])$, $\phi \in C^1[\mathbb{R}]$ are given

and y is unknown. We propose to compute an approximate solution to the equation (1) using the F-transform [6,5,3,4,7]. In details, we construct an operational matrix for the Volterra integral equation as a Hadamard product of two matrices, one of which refers to the Volterra operator, and the other to its kernel. By this, the entire equation can be reduced to a simplier form: the system of nonlinear equations with a triangular matrix. This makes the corresponding numerical method for solving the Volterra integral equation efficient and low computational. The supporting statements, including the convergence of the method, and estimate its computational complexity will be provided.

Additionally, in order to guarantee that the nonlinear equation (1) has a unique solution, we show the applicability of the Fixed Point Theorem, which plays a key role in solving many problems in applied mathematics such as neutron transport, population biology, economics, applied mechanics, etc, see [8,2,9].

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Transfer-Stable Aggregation Functions: Applications, Challenges, and Emerging Trends

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The original transfer-stable aggregation functions generalized the arithmetic means to finite chains. The idea of applying these functions was later demonstrated by purchasing several products depending on the quality and price of the products. This paper aims to continue this idea and show other possible applications of transfer-stable aggregation functions. We identify several concerns in various applications and present possible remedies to address these concerns. We show different types of lattices could be used to

compile the assignment of a given application problem. Based on this finding, we can very effectively divide the products into so-called qualitative classes. We conclude that distance-stable lattices are most effective in these applications. Moreover, we also show that the classes better reflect reality using these lattices.

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Construction for uninorms on the unit interval from t-subnorms and/or t-superconorms

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In [1], we have presented a new construction possibility for t-norms. In this contribution we further develop the construction method applying it for uninorms. The construction possibility we will present is a kind of generalization of the ordinal sum construction by Clifford [4]. Clifford's method was generalized for aggregation operators in [3] and a characterization of uninorms with continuous underlying functions (an ordinal sum construction) was presented in [2].

In our contribution, we assume that the underlying operations of uninorms are ordinal sums of t-subnorms and/or of t-superconorms such that 1 and 0 is their respective idempotent element. Ordinal sum possibilities for t-norms and t-conorms whose summands are t-subnorms and t-superconorms, respectively, will be discussed, and constructions for uninorms from such underlying functions will be presented.

As we will show, the structure of uninorms with ordinal sums of t-subnorms and/or of t-superconorms as underlying functions, is different from the uninorms with ordinal sums of t-norms and t-conorms as underlying functions. Some of the main results are formulated below.

Theorem 1
Let $e \in]0,1[$ and U be a uninorm with e as neutral element. Assume there exist $x \in]0,e[$ and $z \in]e,1[$ such that the restrictions of U to $[0,x]$ and to $[z,1]$ are isomorphic to a t-subnorm and a t-superconorm, respectively, both of them with no non-trivial idempotent elements. Then all values of U in $[0,e[\times]e,1]$ are from $[0,e[$ or all values are from $]e,1]$.

Theorem 2
Let $e \in]0,1[$ and U be a uninorm with e as neutral element. Assume all constraints of Theorem 1 are fulfilled. Let all values in $[0,e[\times]e,1]$ be from $[0,e[$. Let $a_1 < a_2 < e$ be idempotent elements of U. Then $U(x,y) \in]a_1,a_2[$ for $(x,y) \in]a_1,a_2[\times [e,1]$. Moreover, $U(a,z) = a$ for all $z \geq a$ where $a < e$ is an idempotent element of U.

Concluding our considerations, Theorems 1 and 2 imply that, if $a_1 < a_2 < a_3 < e$ are arbitrary three idempotent elements of U, the values $U(x_1,x_2)$ for $(x_1,x_2) \in]a_2,a_3[\times [a_3,1]$ are not influenced by the values $U(x_3,x_4)$ for $(x_3,x_4) \in]a_1,a_2[\times [a_2,1]$.

Some more properties characterizing uninorms of the above discussed form will be presented. Our aim is to get a complete characterization of this class of uninorms which bears some elements of ordinal sum construction.

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How to help Cinderella sort the values of idempotent uninorms on bounded lattices

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Uninorms defined in the unit interval were introduced by Yager and Rybalov [1] as aggregation functions to simultaneously generalize both t-norms and t-conorms. They have proved to be useful in a wide range of fields like decision making, neural networks, information fusion, subjective evaluations, expert systems, fuzzy systems

modelling, pseudo-analysis and measure theory, fuzzy DI-subsethood measures and image processing, fuzzy logic, approximate reasoning, etc (see Section 5 in [2] for details). In literature, the most prominent classes of uninorms are the class of uninorms in $\mathscr{U}_{\min} \cup \mathscr{U}_{\max}$, the class of uninorms with continuous underlying functions and the class of idempotent uninorms. The partial continuity is a staple assumption for characterizing first two classes of uninorms. However, characterizing idempotent uninorms is of pure algebra and does not rely on the (partial) continuity.

Generalization of conjunctors in many-valued logics leading to extension of triangular norms from the unit interval to bounded posets and lattices has initiated extensive research of aggregation functions [4,5]. Similarly, in decision problems, several generalizations of means, including means on more general spaces, were considered. Aggregation in these new spaces brought many open problems, e.g., we can mention characterizations of Sugeno integral or special aggregation functions as t-norms, nullnorms, uninorms, n-uninorms on bounded lattices, or various types of construction methods for aggregation functions on lattices and posets, or generating aggregation functions by means of clones on bounded lattices.

In this contribution, we will extend the results of [3] and we will show the complete characterization of idempotent uninorms defined on lattices with at most a single point incomparable with the neutral element. Using a special function that groups incomparable elements, we will divide any bounded lattice into sets with similar properties and then show that each idempotent uninorm defined on a bounded lattice, where each point is comparable with the neutral element, i.e., $I_e = \emptyset$, is just the ordinal sum of these sets. Our task in this case is similar to Cinderella's task, as we have to collect the points that belong together and separate them from the others. In the case when $I_e = \{w\}$, the situation is much more complicated. Using a similar decomposition we will show the complete characterization of several special classes of idempotent uninorms, covering internal uninorms, idempotent uninorms with annihilator w and others. By composition of these special cases we will obtain the complete characterization of all idempotent uninorms defined on a bounded lattice with exactly one point incomparable with the neutral element. Finally, we will discuss idempotent uninorms defined on even more complex bounded lattices.

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Clifford ordinal sum yeilding a pseudo-uninorm with continuous underlying function

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In this contribution authors aim to present the results of their study on the pseudo-uninorms with continuous underlying functions. A pseudo-uninorm is an associative binary operation with a neutral element $e \in [0,1]$ that is non-decreasing in both coordinates.

The results are concerned mainly on the points of non-commutativity and discontinuity, the possibility of a redefinition of the operator to a commutative uninorm and finally the complete characterization of pseudo-uninorms with continuous underlying functions. It turns out that the points of non-commuatitvity (x, y) of a pseudo-uninorm with continuous underlying functions P may occur only on the graph of its characterizing set-value function. Moreover, in such a case x and y must be idempotent points of the pseudo-uninorm P. At last, the set of semigroups constituting decomposition of a pseudo-uninorm via Clifford ordinal sum is composed of the following semigroups:

- nilpotent t-norm on $[a, b[$
- nilpotent t-conorm on $]a, b]$
- strict t-norm on $]a, b[$
- strict t-conorm on $]a, b[$
- representable uninorm on $]a, b[\cup\{v\}\cup]c, d[$
- trivial semigroups on $\{a\}$
- projections on the first (or last) coordinate on $\{a, c\}$,

where $a < b < e < c < d$ and $v \in [b, c]$.

This contribution extends similar results for uninorms with continuous underlying functions shown in [1] and follows the characterization of idempotent pseudo-uninorms [2] and pseudo-uninorms with continuous Archimedean underlying functions [3].

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Some aspects of the generalization of probability

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In our contribution, we would like to deal with the issue of big data and generalized probability from several points of view. Many big data files are also loaded with random events. Therefore it is necessary to take this feature into account when processing them. Currently, there are several diametrically different approaches to data processing, e.g., [2,3]. It is not correct to apply only the classical theory of probability (probability measure defined on the Boolean σ-algebra). We will focus on the modelling the randomness, which is studied in, e.g., [1,5,4].

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Weak bisimulations: Trading the impreciseness for the finiteness

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Weak bisimulations are a celebrated notion used to compare the behaviors of various fuzzy structures, such as fuzzy labeled transition systems, fuzzy Kripke models, and fuzzy automata. Weak bisimulations have a natural and elegant definition for the latter, as they relate fuzzy sets of accessible and co-accessible states of fuzzy automata (see [3] for more information). From the mathematical aspect, they are just fuzzy relations that are solutions to specific linear systems of fuzzy relation equations or inequalities. Nevertheless, such systems may contain an infinite number of equations or inequalities when the truth value structure is locally non-finite, as in the case of the product structure (cf. [2] for more information). In that case, we demonstrate that we can truncate the product structure with the support $[0, 1]$ to a new structure with the support $[\varepsilon, 1]$, where $\varepsilon \in (0, 1)$ is some small value (a threshold). With the newly defined operators of multiplication and residuum, we show that this new structure, which we call the ε-truncated product structure, is still a complete residuated lattice. However, this structure is now locally finite for any threshold ε. Thanks to it, the given linear systems have a finite number of fuzzy relation equations or inequalities over the ε-truncated product structure. Consequently, if these systems admit a solution, a weak bisimulation can always be computed for fuzzy automata under that structure. However, the price to pay is that we only keep the equivalence of fuzzy automata in the product structure to a certain degree. Nevertheless, this degree is not greater than ε, and the lower the threshold ε, the better the approximation of behavior equivalence is achieved.

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Similarity Functions and Medoids for Honeycomb-Based Structures

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Honeycomb-based structures are prevalent in various scientific fields, such as mathematics, chemistry, and physics. Their frequent occurrence and wide-ranging applications necessitate the development of more efficient methods for processing and analysis. As a result, researchers have been actively pursuing more efficient methods for processing and analyzing these structures.

We present new aggregating methods for honeycomb-based structures. Our approach involves representing such structures using binary sequences, departing from the traditional treatment of these structures as oriented or labeled graphs. This innovative representation streamlines research while preserving essential structural information. We extend the idea of representing the honeycomb-based polygonal chains with binary sequences to the general case. It empowers us to examine pre-existing sequence and string similarity functions (e.g., Hamming distance) while innovating and introducing novel ones. Moreover, new binary operators contribute to the creation of innovative methods for aggregating these structures and identifying related medoids.

We analyze the advantages and disadvantages of each considered similarity function when used as a comparison operator and as a medoid generator. Furthermore, we illustrate each case with an example involving related structures.

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The general algebraic solution of dual fuzzy linear systems

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In [3], a general algebraic solution of fuzzy linear systems of Friedman et al.'s type has been characterized, and the algorithm based on the Moore-Penrose inverse of coefficient matrix for

obtaining the general (strong) solution of non-square FLS has been presented. The straightforward method recently proposed in [1] for solving $m \times n$ FLS $A\tilde{X} = \tilde{Y}$ is a generalization of the obtained results from [3,2]. The presented method provides the possibility to choose any of $\{1\}$-inverses of the coefficient matrix A, not necessarily the Moore-Penrose inverse, since it is based on the general algebraic (strong) solution form of $m \times n$ FLS.

In this paper, a method for solving a dual fuzzy linear system (DFLS), $A\tilde{X} + \tilde{C} = B\tilde{X} + \tilde{D}$, is obtained, where the coefficient matrices $A, B \in \mathscr{M}^{m\times n}$ are arbitrary real $m \times n$ matrices and $\tilde{C}, \tilde{D} \in \mathscr{V}_m$ are given fuzzy number vectors. A necessary and sufficient condition for the $\mathscr{R}$-consistency of the associated system of linear equations is obtained, related to its representative solutions. Moreover, the general form of representative solutions of such linear systems is presented as follows

$$\mathscr{AIS}^{\mathscr{R}} = \Big\{ X \in \mathscr{F}^R \,|\, X = X^* + \frac{1}{2}\mathsf{L} + \mathsf{T}, \\ (S_A - S_B)\mathsf{L} = \mathsf{O}\ (S_A - S_B)\mathsf{T} = \mathsf{R} \Big\},$$

where $X^* = S_G(D - C)$ for some $G \in (A - B)\{1\}$ and $\mathsf{R} = D - C - (S_A - S_B)X^*$. The straightforward method for solving $m \times n$ DFLS based on an arbitrary $\{1\}$-inverse of $A-B$ is introduced. This method is illustrated by interesting examples. Also, as an application, we present a new method for solving a wide class of fuzzy matrix equations based on $\{1\}$-inverses of involved real matrices.

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Fuzzy Association Rules on Data using Intermediate Quantifiers

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In the real world, interpretation, understanding, and deriving information using natural language is prevalent. Particularly, fuzzy sets have been proven successful in linguistically summarising numerical data and extracting knowledge in natural language from large datasets in the form of e.g. fuzzy association rules [1,2] or linguistic summaries [3,4,5]. In our contribution, we focus on implicative rules, which belong to the class of fuzzy association rules and have been proven an efficient tool for the explanatory analysis of large data sets.

In this work, we design fuzzy associative rules for economic time series databases describing the analysis of the expectations of customers towards inflation vs. the sentiment of communication of central banks. The main tool for language interpretation is the GUHA method [5] and further generalized intermediate quantifiers, e.g., “Almost all”, “A few”, “Several”, “Most”, etc.

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A First Approach Toward Diagnosing Mood Disorders from Incomplete Data By Means of Fuzzy Natural Logic

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Psychiatric diagnosis relies heavily on interviewing the patient [4]. The vagueness of the language used to detect a symptom increases as the subjectivity of the symptom description increases. This fact leads to greater uncertainty regarding the diagnosis. Two symptoms of a major depressive episode are abnormal psychomotor activity and amount of sleep that are most often detected using the perceptions of the patient him/herself, people close to the patient or a health professional [6].

Our objective is to apply a computational method whose results can be understood by psychiatry and psychology professionals to a real problem of diagnosing major depressive and bipolar disorders using data obtained with actigraphy [7]. Based on these considerations, we opted for fuzzy natural logic since this theory takes linguistic expressions [2] and human reasoning into account and seeks logical conclusions that are applicable to complicated and real situations, thus being a resource that helps to develop diagnostic reasoning and not replace it.

We created fifteen implicative rules and designed fuzzy sets that describe the total sleep time and motor activity of each participant in terms of evaluative linguistic expressions [2,5]. We extracted information from an actigraph by analyzing the information present in the first seven days of using the device. Table 1 shows the percentage of participants in each extension of expressions describing the state of health as proposed in [3].

Group	very well	well	borderline	unwell	very unwell
Condition	8.69%	13.04%	34.78%	26.09%	17.39%
Control	53.12%	12.5%	21.87%	12.5%	0%

Table 1: Percentage of individuals in a group belonging to each extension of linguistic expressions evaluating the consequents of the rules. The rules chosen to evaluate each participant are in accordance with the Perception-Based Logical Deduction method [5].

In order to increase the number of patients identified as unhealthy and reduce the number of

rules, we resorted to the theory of intermediate quantifiers [1] and generated three implicative rules that evaluate if the patient is unwell or very unwell over a time frame of $7-14$ days, depending on data availability:

- $\mathscr{R}_1$: If motor activity is **not high almost every day** AND total sleep time is **high or very low most days**, then the participant is **very unwell**.
- $\mathscr{R}_2$: If motor activity is **low most days**, then the participant is **very unwell**.
- $\mathscr{R}_3$: : If motor activity is **not high almost every day** OR total sleep time is **high or very low most days**, then the participant is **unwell**.

State of Health	Condition	Control
Very Unwell	17.39%	0%
Unwell	43.48%	18.75%
Total	60.87%	18.75%

Table 2: Labels and average degrees of belonging to each group according to the chance of having a mood disorder, using intermediate quantifiers.

In this paper, we only considered two of the nine symptoms of a major depressive episode [6]. Nevertheless, we succeeded in classifying 43.48% of the condition group as unwell or very unwell using fifteen implicative rules in Table 1. Although we reduced the number of rules to merely three, we were able to identify 60.87% of the condition group as unhealthy, i.e., unwell or very unwell, using intermediate quantifiers as shown in Table 2. Due to restrictions in the number of words (we are already far beyond the limit), we have to postpone the description of the details of these experiments. We also intend to seek the guidance and collaboration of a professional in the area.

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Conversion in Fuzzy Natural Logic

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People commonly use quantifiers in the real-life conversation. In this talk, we will deal with quantifiers of type $\langle 1, 1\rangle$. An example of these quantifiers can be the following: *All children like chocolate* where *All* is a quantifier, *children* is a subject and *like chocolate* is a predicate. When we deal with this type of quantifiers we can distinguish two types of inferences - mediate and immediate inferences. Our goal in both cases is to obtain new information (another quantifier). Mediate inferences use more than one quantifier for the inference. On the other hand, immediate inferences use only one quantifier for inference (see [4]).

Some of the immediate inferences are connected with some structure of opposition (see [3, 2]). But there are also other kinds of immediate inference: *conversion, obversion, contraposition.* In this presentation, we will focus on *conversion.*

Conversion interchanges the subject and the predicate of a quantifier and after this interchange, the truth value remains the same. In the classical case, we have two classical quantifiers (all, some) and truth values are {0,1}. We will show immediate inference by conversion in the following example.

Example 1 Let us know that:
"No lions are herbivores ". is TRUE
Then we can infer by conversion:
"No herbivores are lions" is TRUE

The goal of this talk is to generalize this classical case by using fuzzy intermediate quantifiers. These quantifiers were introduced in Fuzzy natural logic by Novák in [1]. Generalization comes in several ways. We distinguish not only classical quantifiers but also intermediate quantifiers (almost all, most, many) which lie between classical quantifiers. We also extend truth values to the interval $[0, 1]$.

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Generalized Syllogisms with Intermediate Quantifiers "A few" and "Several"

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This contribution on further study of logical syllogisms with intermediate quantifiers were introduced from a philosophical point of view first in [5] and then further elaborated in [4]. In our previous publications [3,4], we proved the validity of logical syllogisms with intermediate quantifiers "Almost all", "Most", and "Many". In this presentation, we will introduce new forms of logical syllogisms with new forms of intermediate quantifiers, namely "A few" and "Several". An interesting area for discussion are non-trivial syllogisms of the third figure and the verification of their validity. In our further research we plan to extend generalized Peterson's rules proposed in [4]. The latter can be used for quick verification of the validity of syllogisms without formal proof or construction of a model. Our planned extension should work also for the new forms of syllogisms introduced above.
Keywords: Fuzzy natural logic, Intermediate quantifiers, Intermediate syllogisms, Quantifiers "A few" and "Several"

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Pseudo-grouping functions obtained by the distortion of aggregation functions: an application on text-based Convolutional Neural Networks

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Convolutional Neural Networks (CNN) are a type of neural networks relying on the extraction and combination of spatial features. Their effectiveness has been demonstrated in numerous applications in computer vision, but have also been successful in the field of natural language processing [1]. In all such applications, the extracted features suffer from high redundancy. This inherent redundancy is deal with by pooling layers, which serve to merge the local features to obtain a single local-to-global feature as a representative. For image processing applications, pooling is typically performed with symmetric operators, since visual features are expected to be invariant w.r.t. image rotations or translations. However, in text-based CNNs, features are time-dependent. In other words, the order of the features might have a significant impact in global analysis. Hence, symmetric functions might not be necessarily the best option for feature pooling. Examples of these not necessarily symmetric functions are pseudo-t-conorms [3] and pseudo-grouping functions [1]. In this work, we study the use of pseudo-grouping functions as pooling operators in natural-language oriented CNNs. In order to do so, we first propose constructions methods for pseudo-grouping functions by distorting aggregation functions via pseudo-automorphisms. Pseudo-grouping functions are further used to merge local features in text-based CNNs. Experimentally, we have obtained statistically significant improvements in performance when compared to standard symmetric operators.

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Fusion function combination through an extension of penalty-based functions for feature reduction in Convolutional Neural Networks

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Fusion and aggregation functions are some of the most widely employed tools in Information Fusion. Usually, different functions are best suited for particular machine learning tasks, but when we face a complex problem or lack expert knowledge, the selection of the most suitable one is not always straightforward.

Penalty-based functions were introduced in order to alleviate this difficulty [6] and allow to select the best reduction of a set of values out of a selection of fusion functions. In the past, they have been used with success for tasks such as image downsampling [5] and decision making [7].

An alternative approach to that of selecting the best fusion function, is that of combining different ones through their convex combination. This allows to preserve the information provided by each of them, although the problem of weighting each one according to its importance needs to be addressed. In [4], we made use of a similar approach to replace the feature reduction process of a Convolutional Neural Network (CNN) in its pooling layer, improving the model's behaviour.

In this work, we try to combine the best of both approaches through an extension of penalty-based functions, which we refer to as wPA-functions. Rather than using a penalty function to select the best single function, we propose to use it to evaluate the suitability of each possible function, with smaller penalty values being associated with more important reductions. This information is then used to compute the weight for each term of a convex combination of the available fusion functions. We present construction methods which allow to tune the strictness of the weighting procedure, and study the conditions under which the resulting wPA-functions can be considered preaggregation functions [3]. We also improve the results obtained in [4] by replacing our previous approach by the new wPA-functions in the pooling layer of our CNN model.

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Efficient Use of Large Language Models for Analysis of Text Corpora

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In this paper, we propose an efficient approach for tracking a given phenomenon in a corpus using natural language processing (NLP) methods. The topic of tracking phenomena in a corpus is important, especially in the fields of sociology, psychology, and economics, which study human behaviour in society. Unlike existing approaches that rely on universal large language models (LLMs), which are computationally expensive, we focus on using computationally less expensive methods. These methods allow for high data processing speed while maintaining high accuracy. Our approach is inspired by the cascade approach to optimization, where we first roughly filter out unwanted information and then gradually use more accurate models, which are computationally more expensive. In this way, we are able to process large amounts of data with high accuracy using different models, while also reducing the overall cost of computations. To demonstrate the proposed method, we chose a task that consists of finding the frequency of occurrence of a certain phenomenon in a large text corpus, which is divided into individual months of the year. In practice, this means that we can, for example, use Internet discussions to find out how much people are discussing a particular topic. The entire solution is presented as a pipeline, which consists of individual phases that successively process text data using methods selected to minimize the overall cost of processing all data.

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Early Warning of River Overflow Flooding Using AI

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Floods pose the most frequent type of natural disaster in the word, causing a huge amount of structural damage on populated areas with the destruction of crops, vehicles, properties, etc. and the irreplaceable loss of human lives. All this leading to great economic costs and mental health issues on the citizens.
On the prevention site, it is important to dispose of early warnings of floods for anticipating the events and apply the emergency plan to reduce the possible impact. Floods can be originated from different causes, the most common are by overflow on the river capacity invading residential areas, and severe stationary rains where the water can get stuck on the surface.
The traditional approach for getting these alerts is with the use of physical models that simulates the behaviour of a river by approximating physical laws. However, the use of AI on the nowcasting field is offering significant advantages in terms of efficiency, accuracy, and flexibility, which makes them more appropriate for this application [1].
We propose to use a fuzzy inference system model based on Takagi-Sugeno-Kang approach (TSK) and optimize it using an Adaptative Neuro-Fuzzy Inference System (ANFIS) to predict the river flow for the next 8 hours on the Arga river (Province of Navarra, Spain), which caused some episodes of floodings on the city of Pamplona. This model performs better than the hydrological models currently used, which can only reach up to 2 hours of anticipation time, and improved Long Short-Term Memory (LSTM) model, which previous work considered it as the best AI model for this river due to its capability to learn time related patterns.
The dataset used consist of meteorological variables supplied from the statal agency AEMET and hydrographic variables from the Hydrographic confederation of the Ebro River (CHEBRO).

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A note on the minimax decomposition integrals

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A new modification of decomposition integrals was introduced in the paper [1] by Šeliga et al. Decomposition systems characterize this class of integrals as in the case of the classical decomposition integrals. Note that the classical decomposition integrals are the suprema of collection integrals with respect to collections contained in the chosen decomposition system, the minimax integrals are the infimum.

These integrals have interesting properties. First of all, they are positively homogeneous and

piecewise linear aggregation functions. Moreover, the minimax integral is the super-additive operator. These integrals give a natural rise to a new relation on the class of decomposition systems.

A dual approach, to replace the outer infimum by supremum in the case of super-decomposition integrals leads to a concept of the maximin decomposition integrals. They are sub-additive positively homogeneous piecewise linear aggregation functions.

Both the minimax and the maximin decomposition integrals are exemplified.

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Methods of aggregation process in federated learning models

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Federated learning with respect to their assumptions is the perfect method to solve problems in many aspects of operating businesses.
In many practical aspects data are crucial because many businesses collect and use their data to enhance their performance. However, limited data or low data quality can hinder model development, particularly in dynamic environments. To get better effects in decision processes, companies collecting similar data may opt to exchange knowledge without sharing their data, due to privacy or legal issues. This is why federated learning gives us possibilities to improve by these requirements.
Especially, horizontal federated learning will be explored, where each client (organization) iteratively improves its model, so that it can be regularly aggregated and shared with all clients participating in the federation for further improvements. In federated averaging, the aggregation mechanism is based on the different methods available to each client and depends on them.
In this paper, we propose to use a more advanced aggregation mechanism and we compare it with other methods from adequate literature [6]-[6].

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Uniqueness of generators is what we need

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When studying functions, we often need a rough description of their shapes in terms like increasing, convex, etc. E.g., given a function, we have an idea how its derivative and integral look like. This intuition is missing in the study of associative copulas (or, more generally, triangular norms, abbr. t-norms) and their generators. A rare exception is described in [2,3].

Switching attention to generators is a desirable simplification; instead of a binary function, whose associativity is hardly interpreted in their graphs, we may deal with unary functions. These take the advantage that some well-known binary operations are associative. We use addition, leading to *additive generators*, or multiplication, leading to *multiplicative generators*. We point out that some other associative binary operations can be considered and bring advantages in specific contexts.

A problem is that the generators (of any kind) are not unique; additive generators are determined up to positive multiples, multiplicative generators up to positive powers. This implies that, e.g., the notion of convexity is well-defined for additive generators (if one additive generator is convex, so are all others), but not for multiplicative generators (some of them may be convex, some not, although they generate the same t-norm). A consequence is that knowing a t-norm, we do not have even a rough idea of the "shape" of its generator and vice versa. We wanted to fill in this gap in understanding these notions.

In order to be able to speak of a "shape" of a generator, we need to select a unique one. (Following [5], we denote by t an additive generator and by θ a mulitiplicative one.) We distinguish several possibilities (see also [4]):

1. If the t-norm is nilpotent, its additive generators are bounded and we may normalize the value at 0, requiring $t(0) = 1$.
2. If the t-norm is strict, its additive generators are unbounded. We may require $t(x) = y$ for some fixed $x \in]0,1[$, $y \in]0,\infty[$.
3. We may fix the derivative of an additive generator at 1, $t'(1) = -1$.
4. We may fix the derivative of a multiplicative generator at 1, $\theta'(1) = 1$.
5. We may require $\theta(x) = y$ for some fixed $x, y \in]0,1[$.
6. We may fix the derivative of a multiplicative generator at 0, $\theta'(0) \in]0,\infty[$.

The latter option 6 requires an explanation: If there is such a multiplicative generator, it is unique, as well as the value $\theta'(0)$. We cannot require an equality for this value. We call such a generator *balanced* [6]. We derived several properties relating the "shape" of a balanced generator and the corresponding t-norm [8,7].

The applicability of options 3, 4, 6 is limited by the assumptions that the respective derivatives exist, are non-zero and finite. Nevertheless, they lead to useful comparison of those operations which admit such a description.

Options 2, 5 are universal, but they did not appear much helpful. Of course, the above list is not complete, the ideas can be combined, e.g, by fixing the derivatives at arbitrary points. Also this idea was not much successful.

Our principal message is that the use of derivatives at boundary points allows a progress in understanding the interplay of generators and the corresponding t-norms. We concentrated on the behavior of the operations near the bounds 0 and 1. This is crucial in copulas describing extreme value distributions [9,1].

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Choquet Integral on a Hyperspace

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In paper [2] Boczek et al. introduced a new concept of generalized survival functions and the corresponding conditional aggregation-based Choquet integral based on them. The main ingredient of this construction are conditional aggregation operators that cover many existing aggregations, such as the arithmetic and geometric mean, or famous Choquet, Sugeno and Shilkret integrals.

The original construction of the conditional aggregation-based Choquet integral does not use level sets of real-valued functions. A conditional aggregation operator $\mathsf{A}(\cdot|E)$ assigns a new value $\mathsf{A}(f|E)$ from $[0,+\infty]$ to a function $f : X \to [0,+\infty]$. In order to build a generalized survival function $\mu_{\mathscr{A}}(f,\alpha)$, the family $\mathscr{A}$ of operators $\mathsf{A}(\cdot|E)$ for conditional sets E and a monotone set function μ are used. The generalized survival function becomes a key to introduce conditional aggregation-based Choquet integral of a function f as

$$\mathrm{C}_{\mathscr{A}}(f,\mu) = \int_0^\infty \mu_{\mathscr{A}}(f,\alpha)\,\mathrm{d}\alpha. \tag{1}$$

In our talk we provide a different perspective on the construction of the conditional aggregation-based Choquet integral compared to the original paper [2]. More precisely, a monotone measure space (X,μ) may be assigned a monotone measure space $(\hat{\mathscr{E}},\mathtt{N}_\mu)$ with $\hat{\mathscr{E}} \subseteq 2^X$ and $\mathtt{N}_\mu$ being one of transformations of monotone set functions to the power set, studied by Yager and Mesiar in [1]. Therefore, the generalized Choquet integral (1) can be considered as a standard Choquet integral but on a hyperspace $\hat{\mathscr{E}}$. We show that this presenting of the aggregation-

based Choquet integral simplifies the study of its properties.

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A study on fuzzy planes and its applications in fuzzy plane fitting

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In this paper, I propose different forms of fuzzy planes: a three-point form, an intercept form, and a fuzzy plane passing through a fuzzy point and perpendicular to a given crisp direction. In regression analysis, a plane plays an important role in analyzing the relationships between a dependent and two independent variables, in $\mathbb{R}^3$. However, a crisp plane is not always enough when observed data is inherently imprecise. When description of data is vague, imprecise, or inadequate, their relationship by classical geometry may not be possible. The need for an effective presentation of relation between objects or data whose realizations are inherently imprecise has necessitated the formulation of a fuzzy plane. Hence, I extend the classical definition of a plane to the fuzzy environment. The proposed fuzzy plane is applied to fit a fuzzy plane to the available data sets of imprecise locations in $\mathbb{R}^3$. Moreover, a degree of fuzzily fitted fuzzy plane to the given data sets of imprecise locations is defined. The aim is to provide a unified foundation of a framework for developing fuzzy geometric modeling which will benefit both creative design and computer vision applications.

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Grid-based computing over joint probability distribution

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Due to a large number of diverse classes/construction methods available for modeling joint probability distribution (mainly thanks to decomposition to copula and marginals approach), it may be difficult to use a particular model for general set of purposes such as calculation of probabilities for certain region in variables space and level sets of quantiles, construction of multivariate marginals and conditional distributions, finding moments and other characteristics representing the full joint distribution or its subsystems.

The problem is theoretical (derivation of proper formulas) as well as practical (finding suitable implementation in software packages).

Here we present an approximate yet general solution based on distribution representation via grid of (absolute or relative) frequencies and fast data frame manipulation tools available in widely used R environment. The grid can be generated either directly from density or cumulative distribution function or indirectly from random sample. The solution is scalable both in terms of precision and dimensionality, and is limited only by hardware resources (memory, parallel processing units).

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Interactive computing

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The topic of this contribution belongs to the basic ground of fuzzy set theory. We plan to discuss the notion of interactivity. Although the notion of interactivity is quite old (it has been defined by L. Zadeh in 1975), many mathematicians do not consider it when applying some results of fuzzy set theory. This fact is somewhat surprising because the well-known Zadeh's (extension) extension is a special case of the interactivity-based (sup-J) extension principle.

The notion of interactivity between two or more fuzzy numbers is strongly connected with notions of a joint possibility extension J and a so-called sup-J extension principle, and it has been intensively studied in the last two decades. For instance, in 2004, Fullér, Carlsson, and Majlender ([2]) introduced a relation between interactivity and a joint possibility extension, practical aspects of interactive computing (the one given by a sup-J extension principle) were studied by K. Scheerlinck, B. de Baets, et al. ([1]) about ten years ago, and numerous mathematicians studied also interactive fuzzy arithmetic in the last decade. Surprisingly, the interactive arithmetics can provide some interesting features, which are not available for "standard" fuzzy arithmetic (i.e. the one using Zadeh's extension principle), and the group around E. Esmi, de Barros et al. recently showed useful practical impacts of the interactive computing. For instance, one can mention the existence of the inverse element for the interactive addition, the existence of the interactive derivative, etc.

In our talk, we would like to go further in this direction by studying some aspects of interactive arithmetics: for instance, by studying conditions, under which the interactive operation preserves inverse elements of given operations.

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Differential entropy in the context of fuzzy measures

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Entropy is a concept of high importance in many fields of study. It can be seen as a similarity measure evaluating the resemblance between two distribution functions (thus measures) or in the principle of maximum entropy [3] as an approximation tool for distribution function in situations where uncertainty is included. Generalising (not only) these ideas for the nonadditive setup, a corresponding formula for fuzzy differential entropy is necessary. As in [1], the transition to fuzzy measures instead of probabilities, Choquet integrals (both symmetric and

asymmetric) replacing Lebesgue integral, and Choquet-Radon-Nikodym derivatives [2] in the place of Radon-Nikodym derivatives needs to be presented. With the resulting formulas, one can also study their properties adopted from the additive case.

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On a bipolar extension of L-fuzzy topologies

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When dealing with an object that has a "crisp" mathematical structure (e.g. a topological space or its subset, a group or its element, et al.) it is usually possible to tell whether this object has a given property or not. On the other hand, in conditions of a fuzzy mathematical structure and the availability of fuzzy logic tools, a situation arises when an object can possess a given property only to a certain extent. A lot of work has been done in this area.

In case of a (fuzzy) topological space and when we are talking about the property of openness of its subset, it seems natural to evaluate the degree of openness as the measure of inclusion of a given (fuzzy) set into its interior. If the space in question is crisp, then this degree is always (regardless of which implicator is used in the definition of this measure) equal to either 1 if the set is open, or 0 otherwise. The situation is different if we consider fuzzy sets and fuzzy topology: the degree of openness of a fuzzy set in an L-fuzzy topological space can be any value $a \in L$. Therefore, both from theoretical considerations and for practical use of L-fuzzy topologies, it seems reasonable to replace the openness property (which is actually a crisp concept) of a fuzzy set by a flexible notion "a fuzzy set open to degree $a \in L$" and thus to obtain a convenient approximation of fuzzy sets in an L-fuzzy topological space.

The purpose of this report is to implement the above idea by presenting a model that allows us to assign to each L-fuzzy subset A of a (Chang-Gougen) L-fuzzy topological space (X, τ) a certain value $\mathscr{T}(A) \in \mathscr{L}$, where $\mathscr{L} = L^+ \times L^-$ is the bipolar extension of the residuated lattice L [4], characterizing the degree to which the L-fuzzy set A possesses the openness property in the space (X, τ). As the result we get an $(L, \mathscr{L})$-fuzzy topology $\mathscr{T} : L^X \to \mathscr{L}$ [1] extending the original L-fuzzy topology $\tau \subseteq L^X$ [5], [2]. We discuss the properties, in particular the functorial ones, of this extension and consider the prospects for adapting our model to the study of other topological properties. Particular, in case when L is a Girard monoid (that is complete residuated lattice $(L, \wedge, \mapsto, 0_L, 1_L)$ is satisfying the double negation law $(a \mapsto 0_L) \mapsto 0_L = a$ for every $a \in L$ [3]) our model becomes especially transparent and allows to get further results. Among other things, in this case we establish a connection between the degrees of openness and the degrees of closedness of L-fuzzy sets, thereby opening up additional tools for working with L-fuzzy topologies.

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A note on fuzzy measures in statistical notions

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It is a known fact that fuzzy measures represent a mathematical concept that is highly adaptable to modeling human behavior and opinions. This flexibility comes from relaxed properties imposed on fuzzy measures, i.e., from "softening" additivity which is crucial for the classical measure. On the other hand, dealing with uncertainty, which is often the case in real-life problems, requires processing non-crisp data, i.e., fuzzy values. Therefore, interesting issues appearing in the recent research are the use of fuzzy measures and corresponding integrals in extending classical statistical notions, as well as statistics for fuzzy values. A possible approach to this problem was considered in [4] where the possibilistic mean value, variance, and covariance for fuzzy numbers were introduced. The generalization of mean values based on fuzzy measures and the general fuzzy integral, was considered in [1]. That approach has been applied to a specific type of fuzzy quantities and allowed the α-cuts, associated with the fuzzy quantity in question, to take different degrees of importance that are modeled by a fuzzy measure. Further, an investigation of the expectation of fuzzy events was presented in [3]. The authors have considered the axiomatical approach, corresponding capacities, as well as the form and properties of the expected value of fuzzy events with respect to different types of nonclassical integrals, including the universal integral. Also, statistical parameters when the uncertainty is expressed by a fuzzy measure, i.e., a concept of monotone expectation that leads to a monotone variance and monotone moments were investigated in [2]. The research presented here aims to extend previous work outlined in [1] while taking into the consideration similarities and differences of several other approaches.

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Functors in Fuzzy Category Theory

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Since the inception of fuzzy set theory great effort has been put forward to find fuzzy analogues of basic concepts and structures of classical mathematics and to work out corresponding theories. However most of these investigations are implicitly (or explicitly) working in classical category theory. The fuzzy aspect of these fuzzy structures shows itself by a certain fuzzy framing of the objects or morphisms of the crisp category. So instead of further developing fuzzy structures within crisp categories, the concept of a fuzzy category was introduced [2].

Fuzzy category theory describes category-like structures in which potential objects and potential morphisms are respectively objects and morphisms only to a certain degree. By relaxing the requirements on object and morphisms, we can obtain models for situations that cannot be directly described using the tools of classical category theory. In our talk we will demonstrate fuzzy categories which contain several well known classical categories as their thresholds.

Continuing the work that has been done on functors in fuzzy categories [1], we further develop the theory until the notion of adjoint functors using the unit and co-unit. We start by defining natural transformations in fuzzy categories, which then gives us the definition of adjoint functors. A sufficient condition is also provided for a fuzzy functor so that it admits a left adjoint with a certain degree.

Several examples, which naturally generalise crisp adjoint functors, will be provided whilst also showing a framework, how using functors it is possible to generate new fuzzy categories.

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On Some Properties of Fuzzy Linguistic Summaries

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In this contribution, we consider some aspects of fuzzy linguistic summaries. They are natural language sentences that help describe and sum up a large set of data clearly and briefly. We discuss the notion of the truth of a fuzzy linguistic summary concerning its essential properties.

We focus on a non-contradiction and a double negation, which make a summary consistent. Primarily, we investigate protoforms of type I, like *"Q y's are P "*, where Q is a linguistic quantifier and P is a summarizer. For such protoforms, we show which fuzzy negations preserve the property of double negation. Therefore, we investigate selected properties of fuzzy sets and fuzzy negations. Furthermore, we study some properties of the linguistic quantifiers whose definitions and features impact the truth values.

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Attribute implications in heterogeneous formal contexts

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In computer science, data science, and applied mathematics, the implications $U \Rightarrow V$ over a set A, $U \subseteq A$, and $V \subseteq A$ have been thoroughly investigated. In Formal Concept Analysis [1], the implications $U \Rightarrow V$ over a set of attributes A are called attribute implications. In a formal context, we can interpret attribute implications by the statement that each object having all attributes from U also has all attributes from V. The notion of fuzzy (graded) attribute implications over a set A of attributes was proposed on complete residuated lattices [2]. A complete axiomatization of logic for reasoning with attribute dependencies that involve grades was investigated by Bělohlávek et al. [3]. The researchers thoroughly investigated other interesting approaches and properties of attribute implications in various generalizations of Formal Concept Analysis [8, 7, 6, 4]. In our paper, we describe attribute implications and their validity in our heterogeneous approach [9]. We explore the relationships between the attribute implications, association rules, and GUHA association rules [5]. Moreover, we present the applications of selected types of attribute implications in various domains.

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On the T-powers of 0 in the invariance property on fuzzy implication functions

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Since its proposal in [3], the invariance property with respect to positive powers of t-norms has become one striking additional property to study on fuzzy implication functions. Several authors have devoted their efforts to this topic achieveing recently the characterizations of all fuzzy implication functions which fullfill the invariance with respect to positive powers of a strict or a nilpotent t-norm (see [2,1]).

However, a thorough analysis of the results of all the published papers will lead to a shocking reveal: none of the results consider the invariance property in the domain where it is originally defined. Recall that a fuzzy implication function I is said to be invariant with respect to positive powers of a continuous t-norm T when $I(x,y) = I(x_T^{(r)}, y_T^{(r)})$ for all $r > 0$ and $x, y \in [0,1]$ such that $x_T^{(r)}, y_T^{(r)} \notin \{0,1\}$. In fact, they consider the less restrictive domain $x, y \in (0,1)$ such that $x_T^{(r)}, y_T^{(r)} \neq 0$, i.e., leaving aside the T-powers of 0 (note that the T-powers of 1 were already not considered in the original definition). This interpretation makes sense since the boundaries of the unit square are avoided on the two sides of the equation, and not only on the left side.

In this paper, the goal is to revisit some of the published results in [2,1] when the original definition of the invariance property is considered. The obtained results indicate that the original definition substantially reduces the number of solutions. Moreover, when other additional properties are imposed, the number of solutions shrinks further when only fuzzy implication functions which are constant almost in the whole unit square are available.

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Implicators on some finite lattices

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A fuzzy implicator on a bounded lattice L is a mapping $I : L^2 \to L$, decreasing in the first and increasing in the second variable, fulfilling the boundary conditions $I(0,0) = I(0,1) = I(1,1) = 1$ and $I(1,0) = 0$, where 0 is the bottom and 1 the top element of L. We study the structure and properties of such families of implicators on some simple finite lattices, including chains, horizontal pastings of chains and a few related ones.

Several additional properties of implications are distinguished, for example the exchange principle, the left neutrality principle, the identity principle, the ordering principle, etc. The authors in [6] deal with the number of discrete implications which satisfy some additional properties.

Another property of our interest is the smoothness. We present its definition on a finite chain $L_n = \{0, 1, 2, \ldots, n\}$.

Definition 1 [3] Let $F : L_n^2 \to L_n$ be a binary operator. It is said that F is $k-$smooth (or simply smooth when $k = 1$) if it is $k-$smooth in each argument; that is,

$$|F(x+1, y) - F(x, y)| \leq k,$$

for all $x \in L_n \setminus \{n\}$ and $y \in L_n$, and

$$|F(x, y+1) - F(x, y)| \leq k,$$

for all $x \in L_n$ and $y \in L_n \setminus \{n\}$.

In [5] the authors studied implications based on smooth t-norms and t-conorms and in [4] smooth implications on a finite chain are studied. We extend these results for some finite lattices.

In this direction we develop the results on the sets of t-norms on lattices, achieved in [2] about estimation the number of t-norms on some special type of lattices. Some of them are in the Figure 1. Based on these we construct implicators on lattices.

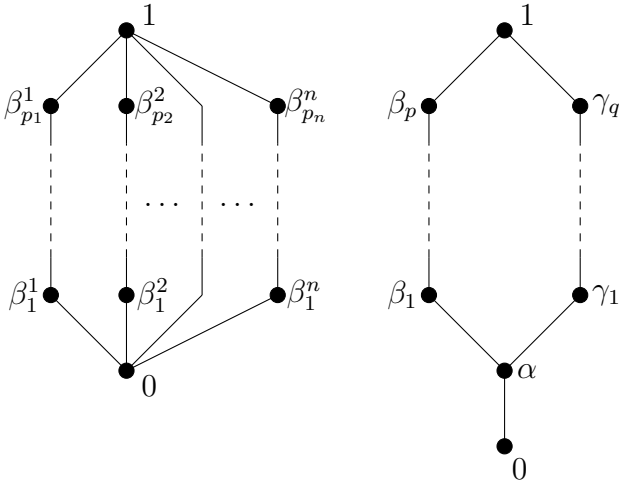

Figure 1: Lattices examples

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